



Professor F. Grillot

Problem set 4

EE 270 - Applied Quantum Mechanics

Due Wednesday Dec. 13, 2017 at 10.00 AM

Prerequisites : For exercises II, III, and IV, read supplementary informations 3 and 4. You may also want to go over the corresponding reading Griffiths, chapters. 6.1, 6.2, 8, 9.1, and 9.2.

Exercise I (10 points)

Compute the expectation values of $\langle r \rangle$ and $\langle r^2 \rangle$ for the ground state of the hydrogen atom (recall that the $|100\rangle = R_{10}(r)Y_{00}(\theta, \varphi)$).

Hint :

$$\int_0^{+\infty} x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

Exercise II (20 points)

Consider an electron in a 1D quantum well of width $L=10$ nm with infinitely high barriers at $(x = \pm L/2)$, under application of an electric field in the x -direction of strength \mathcal{E} . Use an effective mass of $0.067 \times m_0$. We could imagine using the transitions inside this quantum well for a infrared radiation detector.

(a) Calculate the second order perturbation to the energy for the $n = 1$ and $n = 2$ states as a function of field strength. If we measure the absorption of light caused by electrons making transitions from $n = 1 \rightarrow 2$, plot the peak photon energy absorbed versus electric field for field strengths from 0 to 10^7 V/m.

Hint :

$$\int_{-\pi/2}^{+\pi/2} u \sin(ku) \cos(nu) du = \left[\frac{(-1)^{(n+k-1)/2}}{(n+k)^2} - \frac{(-1)^{(n-k-1)/2}}{(n-k)^2} \right]_{n+k=\text{odd}}$$

(b) Estimate, for what range of field strengths might we expect perturbation theory to provide a reasonable approximation.

Exercise III (20 points)

Consider the 2-D simple harmonic oscillator with Hamiltonian

$$\hat{H}(x, y) = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2).$$

Since this Hamiltonian is separable in x and y , the eigenstates are separable into product states $|n_x, n_y\rangle = |n_x\rangle \otimes |n_y\rangle$ with eigenenergies $E_{n_x, n_y} = E_{n_x} + E_{n_y} = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega = (n_x + n_y + 1)\hbar\omega$ and eigenfunctions $\phi_{n_x, n_y}(x, y) = \phi_{n_x}(x)\phi_{n_y}(y)$ where $\phi_{n_x}(x)$ and $\phi_{n_y}(y)$ are eigenfunctions of the 1-D harmonic oscillator.

Suppose we add a perturbation $\hat{H}' = \frac{1}{2}K'\hat{x}\hat{y}$ to the system. How does the three-fold degenerate energy $E = 3\hbar\omega$ split due to this perturbation? To first order, what are the new eigenenergies and eigenstates?

(Hint : Recall that the position operators can be written in terms of the creation/annihilation operators $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_x + \hat{a}_x^\dagger)$ and $\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_y + \hat{a}_y^\dagger)$. Note that each operator only works on the corresponding part of the product state, e.g., $\hat{a}_x|n_x, n_y\rangle = \sqrt{n_x}|n_x - 1, n_y\rangle$, $\hat{a}_x^\dagger|n_x, n_y\rangle = \sqrt{n_x + 1}|n_x + 1, n_y\rangle$, etc.)

Exercise IV (20 points)

Time-dependent perturbation theory implies that the state $|\psi(t)\rangle = (c_a(t), c_b(t))$ of a two-level system with energies E_a and E_b exposed to a time-dependent Hamiltonian H' can be determined exactly by solving for the coefficients

$$\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_{ba}t}c_b \text{ and } \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_{ba}t}c_a$$

where $\omega_{ba} = (E_b - E_a)/\hbar$. In general this problem is not analytically solvable for a sinusoidal interaction $H = 2V \cos(\omega t)$. However, if ω is close to ω_{ba} , we can make the rotating wave approximation (RWA) that $H \simeq V e^{-i\omega t}$, such that $H'_{ba} = V_{ba} e^{-i\omega t}$ and $H'_{ab} = V_{ab} e^{i\omega t}$.

a) Find the exact solution for $|\psi(t)\rangle$ within the RWA for the initial condition $c_a(t = 0) = 1$, $c_b(t = 0) = 0$. Express your results in terms of the Rabi frequency.

$$\omega_r = \frac{1}{2} \sqrt{(\omega - \omega_{ba})^2 + (2|V_{ab}|/\hbar)^2}$$

(Hint : find a differential equation purely in terms of c_b and look for solutions of the form $\exp(\lambda t)$.)



FIGURE 1 – How do Alice and Bob have a "chat" without transferring physical particles ? (Courtesy : John Richardson)

- b) Determine the transition probability $P_{ab}(t)$ and show that it never exceeds 1. Check that the exact $P_{ab}(t)$ reduces to $P_{ab} = \frac{4|V_{ab}|^2}{\hbar^2(\omega_{ba}-\omega)^2} \sin^2 \left[\frac{(\omega_{ba}-\omega)t}{2} \right]$ when the perturbation is small, and state precisely what small means in this context as a constraint on V .
- c) At what time does the system first return to its initial state ?

Exercise V (20 points)

In quantum computing, a quantum bit (qbit) is a unit of quantum information, the quantum analogue of the classical bit. A qbit is a two-state quantum-mechanical system, such as the polarization of a single photon : here the two states are vertical polarization and horizontal polarization. In a classical system, a bit would have to be in one state or the other. However, quantum mechanics allows the qbit to be in a superposition of both states at the same time, a property that is fundamental to quantum computing. A classical bit of information has states 0 or 1 which in quantum mechanics corresponds to the orthonormal basis $|0\rangle$ and $|1\rangle$.

- a) Considering an Hilbert space \mathcal{H} of dimension 2 and $\mathcal{B}_C = \{|0\rangle, |1\rangle\}$ the corresponding computational basis, give the expression of the state vector $|\psi\rangle \in \mathcal{H}$ associated to a qbit.
- b) Show that the probability to obtain the result b is given by :

$$\mathcal{P}(b) = \langle \psi | \hat{P}_b | \psi \rangle \quad b \in \{0, 1\} \quad (1)$$

Write the 2x2 matrices associated to \hat{P}_0 et \hat{P}_1 .

- c) The wavefunction of a quantum system $|\Psi\rangle_{AB}$ can be described as a tensor product of Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$. Now consider two qbits, A et B from the two

Hilbert spaces \mathcal{H}_A and \mathcal{H}_B with the corresponding basis $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ respectively. Determine the basis representing the two qbits operating in $\mathcal{H}_A \otimes \mathcal{H}_B$. Give the expression of a the 2 qbit quantum state.

d) An entangled system is defined to be one whose quantum state cannot be factored as a product of states of its local constituents; that is to say, they are not individual particles but are an inseparable whole. Show that the quantum state of the 2 qbits $|\psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B]$ is entangled.

e) The state $|\psi^+\rangle_{AB}$ can be seen as a state of two particles that is to stay the quantum state of a two-photon system for which each photon is associated to a qbit. We perform a measurement on the first photon in the computational basis. What is the probability to get 0, respectively 1? Now consider a correlation measurement for which each photons is measured independently in the computational basis. What are the possible results of such a measurement?

Exercise VI (10 points)

Consider a system with two electrons (for example two electrons in a box or an Helium atom) in an excited state where the electron configuration is described by placing one of the electrons in a spatial orbital ϕ_a and the other in a spatial orbital ϕ_b .

(a) The possible wavefunctions (and thereby the possible states) of the two electron system can be obtained by writing down first the possible symmetric and antisymmetric spatial wave wavefunctions for the two electrons, the possible symmetric and antisymmetric spin functions of the system (by assigning each electron to either α and β spin "functions") and then considering the possible allowed products of the spatial and spin functions. Remember, that the overall wavefunction for electrons has to be antisymmetric under interchange of electron labels. How many possible states are there and what are the corresponding wavefunctions?

(b) Write down the possible Slater determinants that can be formed from the various simple product spin orbitals such as $\phi_a\alpha$, $\phi_a\beta$, etc.

(c) How are the Slater determinants obtained in part (b) related to the antisymmetric wave functions obtained in part (a) (recall : the Slater determinants are the basic building blocks for forming many electron wavefunctions, find what linear combinations of the Slater determinants in (b) give you the singlet and triplet state wavefunctions).